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A proposal question by **Sunil Kishanchandani**

Let a, b and $c \in \mathbb{R}$, such that $a^2+b^2+c^2=1$. Prove that

$$\frac{1}{5-6bc}+\frac{1}{5-6ca}+\frac{1}{5-6ab} \le 1.$$

Solution by Arkady Alt, **San Jose**,**California**, **USA**.

First note that $5-6bc = 5(a^2+b^2+c^2)-6bc = 5a^2+2(b^2+c^2)+3(b-c)^2 =$ $3a^2+1+3(b-c)^2>0$ and cyclic we have $5-6ca, 5-6ab$.

Furthermore, since $|b|\cdot|c| \ge bc \iff 5-6|b|\cdot|c| \le 5-6bc \iff$ $\dfrac{1}{5-6|b|\cdot|c|} \ge \dfrac{1}{5-6bc}$ (numbers $5-6|b|\cdot|c|, 5-6bc$ are positive) then $\sum \dfrac{1}{5-6bc} \le \sum \dfrac{1}{5-6|b|\cdot|c|}$ and, therefore, inequality of the problem suffices to prove for $a,b,c \ge 0$ (because $\sum \dfrac{1}{5-6bc}$ is invariant with respect to transformation $(a,b,c) \mapsto (-a,-b,-c)$.

Noting that $1-\left(\dfrac{1}{5-6bc}+\dfrac{1}{5-6ca}+\dfrac{1}{5-6ab}\right) =$
$\dfrac{2(25+72abc(a+b+c)-45(ab+ac+bc)-108a^2b^2c^2)}{(5-6bc)(5-6ac)(5-6ab)}$

and, taking in account that $(5-6bc)(5-6ac)(5-6ab)>0$ we can reduce the problem to the proof of inequality

(**1**) $25+72abc(a+b+c)-45(ab+ac+bc)-108a^2b^2c^2 \ge 0$

for any $a,b,c \ge 0$ such that $a^2+b^2+c^2=1$.

After homogenization inequality (**1**) becomes

(**2**) $25(a^2+b^2+c^2)^3+72abc(a+b+c)(a^2+b^2+c^2)-$
$45(ab+ac+bc)(a^2+b^2+c^2)^2-108a^2b^2c^2 \ge 0.$

Using in (**2**) new normalization by $a+b+c=1$ and denoting $p := ab+bc+ca$, $q := abc$ we can rewrite (**1**) as

$25(1-2p)^3+72q(1-2p)-45p(1-2p)^2-108q^2 \ge 0 \iff$

(**3**) $25-380p^3+480p^2-195p+36(2q(1-2p)-3q^2) \ge 0.$

Since $0 \le p \le \dfrac{1}{3}$ then $\dfrac{1-2p}{3} > \dfrac{p}{9} \ge q$ and, therefore, $2q(1-2p)-3q^2$ increase in $q \in \left[0,\dfrac{p}{9}\right]$. Hence, $2q(1-2p)-3q^2 \ge 2q_*(1-2p)-3q_*^2$ where $q_* = \min\left\{0,\dfrac{(1-2t)(1+t)^2}{27}\right\}$ (this value for minimal q give us criteria of solvability Vieta's System $\begin{cases} a+b+c=1 \\ ab+bc+ca=p=\dfrac{1-t^2}{3},\ t\in[0,1] \\ abc=q \end{cases}$ in real $a,b,c \ge 0$.

Since $q_*=0$ for $t \in [1/2,1] \iff p \in [0,1/4]$ then for such p we have
$25-380p^3+480p^2-195p+36(2q(1-2p)-3q^2) \ge 25-380p^3+480p^2-195p =$ $5(5-19p)(2p-1)^2 \ge 0.$

Since $q_* = \dfrac{(1-2t)(1+t)^2}{27}$ for $t \in [0, 1/2] \Leftrightarrow p \in [1/4, 1/3]$ then for such t we have $25 - 380p^3 + 480p^2 - 195p + 36(2q(1-2p) - 3q^2) \geq$

$$25 - 380\left(\frac{1-t^2}{3}\right)^3 + 480\left(\frac{1-t^2}{3}\right)^2 - 195 \cdot \frac{1-t^2}{3} +$$

$$72 \cdot \frac{(1-2t)(1+t)^2}{27}\left(1 - 2 \cdot \frac{1-t^2}{3}\right) - 108 \cdot \left(\frac{(1-2t)(1+t)^2}{27}\right)^2 =$$

$\dfrac{1}{27}t^2(5 - 14t + 26t^2)(3 + 2t + 14t^2) \geq 0$ because

$5 - 14t + 26t^2, 3 + 2t + 14t^2 > 0$ for any t.